Quiz 1

MTH 403, Semester 1, 2021-22

Duration: 50 minutes

Maximum Points: 30

Directions: While writing solutions, please ensure that you sufficiently explain and justify all intermediate arguments leading to any conclusions that you may draw along the way. Each statement (or argument) in your solution must be clearly explained, and must be devoid of any logical fallacies or gaps.

1. For a closed subset $S \subset \mathbb{R}^n$, we define the *distance function* by

$$\text{dist} : \mathbb{R}^n \to \mathbb{R} : x \overset{\text{dist}}{\longmapsto} \inf\{|x - s| : s \in S\}.$$

Let $E(S)$ be the set of all $x \in \mathbb{R}^n$ satisfying the condition that if there exists $s_1, s_2 \in S$ such that

$$|x - s_1| = |x - s_2| = \text{dist}(x, S),$$

then $s_1 = s_2$. Now define the *nearest point function* by $\xi : E(S) \to S$ by $\xi(x) = y$, where $y \in S$ is a unique point satisfying $|x - y| = \text{dist}(x, S)$. [5+10+5]

(a) Show that the function ξ is continuous.

(b) Given an $x \in E(S) \setminus S$ and a unit vector v, show that

$$\text{dist}'(x; v) = v \cdot \frac{x - \xi(x)}{\text{dist}(x)},$$

whenever it exists.

(c) Show that the function dist is of class C^1 on $(E(S) \setminus S)^\circ$.

2. A homeomorphism $f : A(\subset \mathbb{R}^n) \to \mathbb{R}^n$, where A is a open, is said to be a *diffeomorphism* if both f and f^{-1} are differentiable. Let $f : \mathbb{R}^2 \to \mathbb{R}^1$ be of class C^1. Consider a point $y \in f(\mathbb{R}^2)$ such that at each $x \in f^{-1}(y)$, $Df(x)$ is surjective. Then show that $f^{-1}(y)$ is locally diffeomorphic to $\mathbb{R}^1$. (Hint: Use the Inverse function theorem.) [10]